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# Solutions of Second Order Ordinary Differential Equations*

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## 1. INTRODUCTION

We assume that $f(x, y, y')$ is defined and continuous either on

$$R = \{(x, y, y') : a \leqslant x \leqslant b, |y| + |y'| < +\infty\},$$

$$S = \{(x, y, y') : a \leqslant x < +\infty, |y| + |y'| < +\infty\},$$

or

$$T = \{(x, y, y') : -\infty < x < +\infty, |y| + |y'| < +\infty\}.$$

For such a function $f$ we will be interested in the differential equation

$$y'' = f(x, y, y') \tag{1.1}$$

and will give sufficient conditions for the existence of solutions to the problems

$$y'' = f(x, y, y'), y(a) = \alpha, y(b) = \beta \quad \text{for } f \text{ defined on } R, \tag{1.2}$$

$$y'' = f(x, y, y'), y(a) = \alpha \quad \text{for } f \text{ defined on } S, \tag{1.3}$$

and

$$y'' = f(x, y, y'), \qquad \text{for } f \text{ defined on } T, \tag{1.4}$$

where it is understood that the solutions are to exist on $[a, b]$, $[a, +\infty)$ or $(-\infty, +\infty)$ respectively.

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In recent years a number of authors have addressed themselves to these problems. In particular, Bebernes [1] and Fountain and Jackson [2] have given results for problem (1.2) assuming $f$ is nondecreasing as a function of $y$, while Wong [3] has considered (1.3) where $f$ does not depend on $y'$. In other recent papers Belova [4] has treated (1.3) and (1.4) while Gross [5] has studied (1.3) with the $y'$ variable missing. Schuur [6] has imposed the restrictions that $f$ be nondecreasing in both $y$ and $y'$, $f(x, 0, 0) = 0$ and solutions of (1.1) are uniquely determined by initial conditions.

Jackson and the author in [7], [8] have studied these problems but as in [9], and most of the other papers cited, there is a severe restriction placed on the growth of $f$ in the $y'$ variable. In [8], for example, $|f|$ is not allowed to grow significantly faster than $0((y')^2)$ for $x$ and $y$ restricted to a compact set. It is the purpose of this paper to provide existence theorems for functions $f$ which do not satisfy this restricted rate of growth in the $y'$ variable, and we will do this without requiring that $f$ be nondecreasing in $y$ or satisfy any Lipschitz conditions.

## 2. Preliminary Remarks

For $I$ an interval, $I^0$ the interior of $I$, a function $\phi \in C^2(I^0) \cap C(I)$ will be called a lower solution of (1.1) on $I$ in case $\phi'' \geqslant f(x, \phi, \phi')$ on $I^0$. Similarly, $\psi \in C^2(I^0) \cap C(I)$ will be called an upper solution of (1.1) on $I$ in case $\psi'' \leqslant f(x, \psi, \psi')$ on $I^0$.

In addition to always assuming that $f$ in (1.1) is continuous, we will impose subsets of the following conditions on $f$ as needed. In each case, $I$ is to be a compact interval contained in the $x$ domain of $f$ and $\phi, \psi \in C(I)$ with $\phi(x) \leqslant \psi(x)$ for $x \in I$. Also, $h$ and $k$ are positive continuous functions defined for $t \geqslant 0$ such that

$$\int_0^\infty \frac{tdt}{h(t)} = +\infty = \int_0^\infty \frac{tdt}{k(t)}.$$

(1) For each $I$ there is an $h$ such that $f(x, y, y') \geqslant -h(y')$ for $y' \geqslant 0$, $x \in I$ and $\phi(x) \leqslant y \leqslant \psi(x)$.

(2) For each $I$ there is an $h$ such that $f(x, y, y') \leqslant h(y')$ for $y' \geqslant 0$, $x \in I$ and $\phi(x) \leqslant y \leqslant \psi(x)$.

(3) For each $I$ there is a $k$ such that $f(x, y, y') \leqslant k(-y')$ for $y' \leqslant 0$, $x \in I$ and $\phi(x) \leqslant y \leqslant \psi(x)$.

(4) For each $I$ there is a $k$ such that $f(x, y, y') \geqslant -k(-y')$ for $y' \leqslant 0$, $x \in I$ and $\phi(x) \leqslant y \leqslant \psi(x)$.

## 3. Solutions on $[a, b]$

In this section we assume $f$ is defined on $R$ and will give sufficient conditions for the existence of a solution to problem (1.2).

THEOREM 3.1. *Let $\phi, \psi \in C^1[a, b] \cap C^2(a, b)$ with $\phi(x) \leqslant \psi(x)$ for $x \in [a, b]$, and $\phi, \psi$ are lower and upper solutions, respectively, for* (1.1). *Assume that $f$ satisfies* (1), (2), (3) *and* (4). *Then problem* (1.2) *has a solution $y$ with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, b]$, for any $\alpha$ and $\beta$ satisfying $\phi(a) \leqslant \alpha \leqslant \psi(a)$, and $\phi(b) \leqslant \beta \leqslant \psi(b)$.*

*Proof.* Let $h_1$, $h_2$, $k_1$ and $k_2$ be the functions coming from conditions (1), (2), (3) and (4) respectively. Let

$$\lambda = \max\{|\phi(b) - \psi(a)|/(b - a), |\psi(b) - \phi(a)|/(b - a),$$
$$\times \max_{x\in[a,b]} |\phi'(x)|, \max_{x\in[a,b]} |\psi'(x)|\}$$

and choose $N > 0$ so that

$$N > \max\{M_1, M_2, M_3, M_4\}$$

where

$$\int_\lambda^{M_1} \frac{tdt}{h_1(t)} = \int_\lambda^{M_2} \frac{tdt}{h_2(t)} = \int_\lambda^{M_3} \frac{tdt}{k_1(t)}$$
$$= \int_\lambda^{M_4} \frac{tdt}{k_2(t)} = \max_{x\in[a,b]} \psi(x) - \min_{x\in[a,b]} \phi(x).$$

Define $F(x, y, y')$ by

$$F(x, y, y') = \begin{cases} G(x, y, N) & \text{for} \quad y' > N \\ G(x, y, y') & \text{for} \quad |y'| \leqslant N \\ G(x, y, -N) & \text{for} \quad y' < -N \end{cases}$$

where

$$G(x, y, y') = \begin{cases} f(x, \psi(x), y') + (y - \psi(x))^{1/2} & \text{for} \quad y > \psi(x) \\ f(x, y, y') & \text{for} \quad \phi(x) \leqslant y \leqslant \psi(x) \\ f(x, \phi(x), y') - (\phi(x) - y)^{1/2} & \text{for} \quad y < \phi(x). \end{cases}$$

It follows as in Lemma 2.3 of [7] that problem (1.2) has a solution $z$, with the function $f$ replaced by $F$, and that $\phi(x) \leqslant z(x) \leqslant \psi(x)$ for $x \in [a, b]$. If we can show that $|z'(x)| \leqslant N$, we will be done since $F$ and $f$ agree for $\phi(x) \leqslant y \leqslant \psi(x)$ and $|y'| \leqslant N$.

Let $a < x_0 < b$ be such that $z(b) - z(a) = z'(x_0)(b - a)$; then $|z'(x_0)| \leqslant \lambda < N$. Four cases must be considered depending whether there exists $x_1$ so that $z'(x_1) = N$ or $z'(x_1) = -N$ and whether $x_1 > x_0$ or $x_1 < x_0$. Assume, for example, that $x_0 < x_1$ and that $z'(x_1) = N$, $z'(x_0) = \lambda$ and

$\lambda < z'(x) < N$ for $x_0 < x < x_1$. Now for $x_0 \leqslant x \leqslant x_1$, we have from (2) that $z''(x) \leqslant h_2(z'(x))$ and hence that

$$\frac{z'z''}{h_2(z')} \leqslant z'.$$

If we integrate this inequality from $x_0$ to $x_1$ we obtain

$$\int_\lambda^N \frac{t\,dt}{h_2(t)} \leqslant z(x_1) - z(x_0) \leqslant \max_{x\in[a,b]} \psi(x) - \min_{x\in[a,b]} \phi(x)$$

which contradicts the choice of $N$. Proofs for the other three cases are similar. If $x_1 < x_0$ and $z'(x_1) = N$, condition (1) gives a contradiction. If $x_1 > x_0$ and $z'(x_1) = -N$, condition (4) gives a contradiction, and if $x_1 < x_0$ and $z'(x_1) = -N$, then (3) gives the contradiction.

THEOREM 3.2. *Let $\phi$, $\psi$ be as in Theorem* 3.1 *with $\phi(b) = \psi(b)$, and assume $f$ satisfies* (1) *and* (3). *Then problem* (1.2) *has a solution $y$ with $\phi(x) \leqslant y \leqslant \psi(x)$ for $x \in [a, b]$, for any $\alpha$, $\beta$ satisfying $\phi(a) \leqslant \alpha \leqslant \psi(a)$, and $\phi(b) = \beta = \psi(b)$.*

*Proof.* Let $h$ and $k$ be the functions coming from conditions (1) and (3) respectively. Let $\lambda$ be as in the proof of Theorem 3.1 and choose $N > 0$ so that

$$N > \max\{M_1, M_2\}$$

where

$$\int_\lambda^{M_1} \frac{t\,dt}{h(t)} = \int_\lambda^{M_2} \frac{t\,dt}{k(t)} = \max_{x\in[a,b]} \psi(x) - \min_{x\in[a,b]} \phi(x).$$

With $F(x, y, y')$ as defined before, problem (1.2) has a solution $z$, with the function $f$ replaced by $F$, and $\phi(x) \leqslant z(x) \leqslant \psi(x)$ for $x \in [a, b]$. As before, it suffices to show that $|z'(x)| \leqslant N$. Note that $\phi'(b) \geqslant z'(b) \geqslant \psi'(b)$ so that $|z'(b)| \leqslant \lambda$. Two cases must be considered depending whether there is an $x_1$ so that $z'(x_1) = N$ or $z'(x_1) = -N$. Assume, for example, that $x_1 < x_0$, $z'(x_1) = N$, $z'(x_0) = \lambda$ and $\lambda < z'(x) < N$ for $x_1 < x < x_0$. Using condition (1) as in the proof of Theorem 3.1 yields a contradiction. In the other case, (3) gives a contradiction.

THEOREM 3.3. *Let $\phi$, $\psi$ be as in the proof of Theorem* 3.1 *with $\phi(a) = \psi(a)$ and assume $f$ satisfies* (2) *and* (4). *Then problem* (1.2) *has a solution $y$ with $\phi(x) \leqslant y \leqslant \psi(x)$ for $x \in [a, b]$, for any $\alpha$, $\beta$ satisfying $\phi(a) = \alpha = \psi(a)$, and $\phi(b) \leqslant \beta \leqslant \psi(b)$.*

*Proof.* This proof is similar to the proof of Theorem 3.2, so will be omitted.

THEOREM 3.4. *Let $\phi$, $\psi$ be as in Theorem* 3.1 *and assume f satisfies* (1) *and* (4). *Then problem* (1.2) *has a solution y with $\phi(x) \leqslant y \leqslant \psi(x)$ for $x \in [a, b]$, for $\alpha$, $\beta$ satisfying $\alpha = \phi(a)$, $\beta = \phi(b)$.*

*Proof.* Let $\lambda$, $N$, $F$ and $z$ be as in the proof of Theorem 3.2 where $h$ and $k$ come from conditions (1) and (4). Note that $z'(a) \geqslant \phi'(a)$ and $z'(b) \leqslant \phi'(b)$, so $z'(a) \geqslant -\lambda$ and $z'(b) \leqslant \lambda$. Two cases must be considered depending whether there are $x_0$, $x_1$ with $x_0 < x_1$ so that $z'(x_0) = -\lambda$ and $z'(x_1) = -N$, or $z'(x_0) = N$ and $z'(x_1) = \lambda$. In the first case (4) yields a contradiction, while in the second case (1) does.

THEOREM 3.5. *Let $\phi$, $\psi$ be as in Theorem* 3.1 *and assume f satisfies* (2) *and* (3). *Then problem* (1.2) *has a solution y with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, b]$, for $\alpha$, $\beta$ satisfying $\alpha = \psi(a)$, $\beta = \psi(b)$.*

*Proof.* This proof is similar to the proof of Theorem 3.4, so is omitted.

## 4. SOLUTIONS ON $[a, \infty)$

Here we assume $f$ is defined on $S$ and will give sufficient conditions for the existence of a solution to problem (1.3).

THEOREM 4.1. *Let $\phi, \psi \in C^1[a, \infty) \cap C^2(a, \infty)$ with $\phi(x) \leqslant \psi(x)$ for $x \in [a, \infty)$, and $\phi$, $\psi$ lower and upper solutions, respectively, for* (1.1). *Assume f satisfies* (1) *and* (3) *on every compact subinterval of* $[a, \infty)$. *Then problem* (1.3) *has a solution y with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, \infty)$, for any $\alpha$ satisfying $\phi(a) \leqslant \alpha \leqslant \psi(a)$.*

*Proof.* Let $\{b_n\}$ and $\{\beta_n\}$ be sequences satisfying $b_n = a + n + 2$ for $n \geqslant 1$ and $\phi(b_n) \leqslant \beta_n \leqslant \psi(b_n)$. Then consider the sequence of problems

$$y'' = f(x, y, y'), \qquad y(a) = \alpha, \qquad y(b_n) = \beta_n. \tag{4.1}$$

Let

$$\lambda_n = \max\{(1/2)|\phi(b_n) - \psi(b_n - 2)|, (1/2)|\psi(b_n) - \phi(b_n - 2)|, \max_{x\in[a,b_n]} |\phi'(x)|, \max_{x\in[a,b_n]} |\psi'(x)|\}$$

and choose $N_n > 0$ so that

$$N_n > \max\{M_1, M_2\},$$

where

$$\int_{\lambda_n}^{M_1} \frac{t\,dt}{h(t)} = \int_{\lambda_n}^{M_2} \frac{t\,dt}{k(t)} = \max_{x\in[a,b_n]} \psi(x) - \min_{x\in[a,b_n]} \phi(x)$$

and $h$, $k$ are the functions coming from conditions (1) and (3), respectively, for the interval $[a, b_n]$. If $F_n(x, y, y')$ is defined as before and $z_n$ is the solution to (4.1) [with $f$ replaced by $F_n$] satisfying $\phi(x) \leqslant z_n(x) \leqslant \psi(x)$ for $x \in [a, b_n]$, then we claim that $|z_n'(x)| \leqslant N_n$ for $x \in [a, b_n - 2]$. To see this, note that there is an $x_0$ with $b_n - 2 < x_0 < b_n$ and $z_n(b_n) - z_n(b_n - 2) = 2z_n'(x_0)$, so that $|z_n'(x_0)| \leqslant \lambda$. Two cases must now be considered, depending whether there is an $x_1$ with $a \leqslant x_1 \leqslant b_n - 2$ so that $z'(x_1) = N$ or else $z'(x_1) = -N$. The remainder of this part of the proof is as in the proof of Theorem 3.2.

We have now established that each $z_n$ is a solution of (1.1) on $[a, a + n]$. Moreover, $z_n(a) = \alpha$ for each $n$. We now claim that for each fixed positive integer $m$, all the solutions $z_n$ for $n \geqslant m$ satisfy $|z_n'(x)| \leqslant N_m$ for $a \leqslant x \leqslant a + m$. The proof of this is essentially the same as the proof that $|z_m'(x)| \leqslant N_m$. The remainder of the proof is a standard diagonalization process used to pick a subsequence of $\{z_n\}$ which converges uniformly on each compact interval $[a, a + n]$ and for which the sequence of derivatives converges uniformly. The limit of this subsequence is the desired solution.

THEOREM 4.2. *Let $\phi$, $\psi$ be as in Theorem 4.1 and assume $f$ satisfies (1) and (4) on every compact subinterval of $[a, \infty)$. Then problem (1.3) has a solution $y$ with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, \infty)$ for $\alpha = \phi(a)$.*

*Proof.* Let $\{b_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{N_n\}$ be as in the proof of Theorem 4.1 where $h$ and $k$ are the functions coming from conditions (1) and (4), respectively, for the interval $[a, b_n]$. If $F_n(x, y, y')$ and $z_n$ are as in the proof of Theorem 4.1, then we claim that $|z_n'(x)| \leqslant N_n$ for $x \in [a, b_n - 2]$. To see this, note that there is an $x_0$ with $b_n - 2 < x_0 < b_n$ and $z_n(b_n) - z_n(b_n - 2) = 2z_n'(x_0)$ so that $|z_n'(x_0)| \leqslant \lambda$. Also, $z_n'(a) \geqslant \phi'(a)$ so that $z_n'(a) \geqslant -\lambda$. Two cases are considered depending whether there is $x_1 \in [a, b_n - 2]$ so that $z'(x_1) = N$ or $z'(x_1) = -N$. In the first case (1) gives a contradiction, while in the second case (4) does. We now claim that for each fixed positive integer $m$ the solutions $z_n$ for $n \geqslant m$ satisfy $|z_n'(z)| \leqslant N_m$ for $a \leqslant x \leqslant a + m$. From here on the proof is as in the proof of Theorem 4.1.

THEOREM 4.3. *Let $\phi$, $\psi$ be as in Theorem 4.1 and assume $f$ satisfies (2) and (3) on every compact subinterval of $[a, \infty)$. Then problem (1.3) has a solution $y$ with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, \infty)$, for $\alpha = \psi(a)$.*

*Proof.* This proof is similar to the proof of Theorem 4.2, so is omitted.

THEOREM 4.4. *Let $\phi$, $\psi$ be as in Theorem 4.1 with $\phi(a) = \psi(a)$ and assume $f$ satisfies (2) and (4) on every compact subinterval of $[a, \infty)$. Then problem (1.3) has a solution $y$ with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in [a, \infty)$, for $\phi(a) = \alpha = \psi(a)$.*

*Proof.* With the obvious modifications, this is similar to the proof of Theorem 4.2.

## 5. Solutions on $(-\infty, +\infty)$

In this section $f$ is assumed to be defined on $T$, and conditions will be found which guarantee the existence of a solution to problem (1.4).

THEOREM 5.1. *Let $\phi, \psi \in C^2(-\infty, +\infty)$ with $\phi(x) \leqslant \psi(x)$ for $x \in (-\infty, +\infty)$, and $\phi, \psi$ lower and upper solutions, respectively, for* (1.1). *Assume $f$ satisfies* (1) *and* (3) *on every compact subinterval of $(-\infty, +\infty)$. Then problem* (1.4) *has a solution $y$ with $\phi(x) \leqslant y(x) \leqslant \psi(x)$ for $x \in (-\infty, +\infty)$. The same conclusion follows if we replace* (1) *and* (3) *above by either* (1) *and* (4), (2) *and* (3), *or* (2) *and* (4).

*Proof.* We will only give the proof for $f$ satisfying (1) and (3), as the others are similar. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying $a_n = -n - 2$, $b_n = n + 2$, $\phi(a_n) \leqslant \alpha_n \leqslant \psi(a_n)$ and $\phi(b_n) \leqslant \beta_n \leqslant \psi(b_n)$ for $n \geqslant 1$. Consider the sequence of problems

$$y'' = f(x, y, y'), \qquad y(a_n) = \alpha_n, \qquad y(b_n) = \beta_n. \tag{5.1}$$

Let

$$\begin{aligned}\lambda_n = \max\{&(\tfrac{1}{2})|\phi(b_n) - \psi(b_n - 2)|, (\tfrac{1}{2})|\psi(b_n) - \phi(b_n - 2)|,\\ &(\tfrac{1}{2})|\phi(a_n + 2) - \psi(a_n)|, (\tfrac{1}{2})|\psi(a_n + 2) - \phi(a_n)|,\\ &\max_{x\in[a_n,b_n]} |\phi'(x)|, \max_{x\in[a_n,b_n]} |\psi'(x)|\}\end{aligned}$$

and choose $N_n > 0$ so that

$$N_n > \max\{M_1, M_2\},$$

where

$$\int_{\lambda_n}^{M_1} \frac{tdt}{h(t)} = \int_{\lambda_n}^{M_2} \frac{tdt}{k(t)} = \max_{x\in[a_n,b_n]} \psi(x) - \min_{x\in[a_n,b_n]} \phi(x)$$

and $h$, $k$ are the functions coming from conditions (1) and (3), respectively, for the interval $[a_n, b_n]$. If $F_n(x, y, y')$ is defined as before and $z_n$ is the solution to (5.1) [with $f$ replaced by $F_n$] satisfying $\phi(x) \leqslant z_n(x) \leqslant \psi(x)$ for $x \in [a_n, b_n]$, then we claim $|z_n'(x)| \leqslant N_n$ for $x \in [a_n + 2, b_n - 2]$. For this, note that there are $x_0, x_1$ with $b_n - 2 < x_0 < b_n$, $a_n < x_1 < a_n + 2$, $z_n(b_n) - z_n(b_n - 2) = 2z_n'(x_0)$ and $z_n(a_n + 2) - z_n(a_n) = 2z_n'(x_1)$ so that $|z_n'(x_0)| \leqslant \lambda$ and $|z_n'(x_1)| \leqslant \lambda$. The rest of this proof proceeds similarly to the proof of Theorem 4.2.

## 6. Applications

(a) The existence of the solution which is asserted in Theorem 1.1 [*3*, p. 740] follows from Theorem 4.1, using $\phi = 0$, $\psi = A$, and

$$h = k = 1 + \max_{\substack{x \in I \\ 0 \leqslant y \leqslant A}} yF(y, x).$$

We mention this since the proof as given uses the assumption that $yF(y, x)$ satisfies a Lipschitz condition with respect to $y$ in every closed rectangle

$$\{(x, y) : 0 \leqslant y \leqslant K, a \leqslant x \leqslant b\},$$

which does not follow from the stated conditions as claimed [*3*, p. 737]. This can be seen by examining the function $yF(y, x) = y[1 + (y - 1)^{1/3}]$ for $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 2$.

(b) The existence of the solution which is asserted in Theorem 2 [*6*, p. 596] follows from Theorem 4.1, using $\phi = 0$, $\psi = A$, $h = 1$ and

$$k = 1 + \max_{x \in I} |f(x, A, 0)| .$$

The proof that is given for this theorem depends on a lemma [*6*, p. 596] which is incorrect since the conditions imposed on $f$ do not insure that solutions to initial value problems are extendable.

(c) Although it should be clear that these results apply to problems not covered by the papers cited in the introduction, we remark that the problem

$$y'' = -y + (y')^{2n}, \quad y(0) = y(1) = 1, \tag{6.1}$$

where $n$ is an arbitrary positive integer, is solvable by Theorem 3.4, using $\phi = 1$, $\psi = 2x + 1$, and $h = k = 3$.

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